

## RMO–2002

1. In an acute triangle  $ABC$ , points  $D, E, F$  are located on the sides  $BC, CA, AB$  respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that  $AD, BE, CF$  are the altitudes of  $ABC$ .

2. Solve the following equation for real  $x$ :

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

3. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^2$ ,  $b$  divides  $c^2$  and  $c$  divides  $a^2$ . Prove that  $abc$  divides  $(a + b + c)^7$ .
4. Suppose the integers  $1, 2, 3, \dots, 10$  are split into two disjoint collections  $a_1, a_2, a_3, a_4, a_5$  and  $b_1, b_2, b_3, b_4, b_5$  such that

$$a_1 < a_2 < a_3 < a_4 < a_5 \text{ and } b_1 > b_2 > b_3 > b_4 > b_5.$$

- (i) Show that the larger number in any pair  $\{a_j, b_j\}$ ,  $1 \leq j \leq 5$ , is at least 6.
- (ii) Show that  $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$  for every such partition.
5. The circumference of a circle is divided into eight arcs by a convex quadrilateral  $ABCD$ , with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by  $p, q, r, s$  in counter-clockwise direction starting from some arc. Suppose  $p + r = q + s$ . Prove that  $ABCD$  is a cyclic quadrilateral.
6. For any natural number  $n > 1$ , prove the inequality :

$$\frac{1}{2} < \frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \dots + \frac{n}{n^2 + n} < \frac{1}{2} + \frac{1}{2n}.$$

7. Find all integers  $a, b, c, d$  satisfying the following relations :

- (i)  $1 \leq a \leq b \leq c \leq d$ ;
- (ii)  $ab + cd = a + b + c + d + 3$ .