INMO–2008

1. Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

2. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

3. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

4. All the points with integer coordinates in the xy-plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is coloured red and the point $(0, 1)$ is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

5. Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

6. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (i) $P(x)Q(x)$ is a polynomial in x^2; and (ii) $P(x)R(x)$ is a polynomial in x^3.