INMO–2007

1. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that
$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

2. Let n be a natural number such that $n = a^2 + b^2 + c^2$, for some natural numbers a, b, c. Prove that
$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2,$$
where p_j's, q_j's, r_j's are all **nonzero** integers. Further, of 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

3. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

4. Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of $(1, 2, 3, \ldots, n)$. A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (Example: In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, $(3, 1)$.) How many permutations of $(1, 2, 3, \ldots, n)$, $(n \geq 3)$, have exactly **two** inversions.

5. Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of the perpendicular from P on AC. If $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \lambda$, $\dfrac{BD}{AD} = m$ and $z = m^2(1 + \lambda)$, prove that
$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$
Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

6. If x, y, z are positive real numbers, prove that
$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$