INMO–2004

February 1, 2004

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively . Suppose

   (a) BD bisects KM at Q ;

   (b) $QA = QB = QC = QD$; and

   (c) $LK/LM = CD/CB$.

   Prove that $ABCD$ is a square .

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a , we denote by $[a]$ the greatest integer not exceeding a .)

4. Let R denote the circumradius of a triangle $ABC; a, b, c$ its sides BC, CA, AB ; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

   (i) $a > b$ and $a > c$;

   (ii) $2R > r_b$ and $2R > r_c$.

5. Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

   Find the greatest common divisor of all the members of T.

6. Prove that the number of 5-tuples of positive integers (a,b,c,d,e) satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

   is an odd integer .