INMO–1994

Time : 4 hours.

Answer as many questions as you possibly can.

1. Let G be the centroid of a triangle ABC in which the angle C is obtuse and AD and CF be the medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that
$$\frac{AC}{BC} > \sqrt{2}.$$
If further P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangle ABC and GAP are similar.
2. If $x^5 - x^3 + x = a$, prove that $x^6 \geq 2a - 1$.
3. In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19.
4. Find the number of nondegenerate triangles whose vertices lie in the set of points (s, t) in the plane such that $0 \leq s \leq 4$, $0 \leq t \leq 4$, with s and t integers.
5. A circle passes through a vertex C of a triangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units, find the area of the rectangle $ABCD$.
6. If $f : \Re \to \Re$ is a function satisfying the properties

   (a) $f(-x) = -f(x)$,

   (b) $f(x+1) = f(x) + 1$,

   (c) $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$, for $x \neq 0$,

   prove that $f(x) = x$ for all real values of x. Here $\Re$ denotes the set of all real numbers.