INMO–1992

Attempt as many questions as you possibly can.
Use of calculating aids not permitted.

1. In a triangle ABC, angle A is twice angle B. Show that

$$a^2 = b \cdot (b + c).$$

2. If x, y and z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then show that each of x, y and z lies in the closed interval $[2/3, 2]$, that is $2/3 \leq x \leq 2$, $2/3 \leq y \leq 2$ and $2/3 \leq z \leq 2$. Can x attain the extreme value $2/3$ or 2 ?
3. Find the remainder when 19^{92} is divided by 92.
4. Find the number of permutations $(P_1, P_2, P_3, P_4, P_5, P_6)$ of $1, 2, 3, 4, 5, 6$ such that for any k, $1 \leq k \leq 5$. $(P_1, P_2, \ldots P_k)$ does not form a permutation of $\{1, 2, \ldots k\}$. That is $P_1 \neq 1$; (P_1, P_2) is not permutation of $\{1, 2\}$; (P_1, P_2, P_3) is not a permutation of $\{1, 2, 3\}$, etc.
5. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the mid-point of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the mid-point of XZ.
6. Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for 5 distinct integers a_1, a_2, a_3, a_4 and a_5 one has

$$f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2.$$

Show that there does not exist an integer b such that $f(b) = 9$.

7. Find the number of ways in which one can place the numbers $1, 2, 3, \ldots n^2$ on the n^2 squares of $n \times n$ chessboard, one on each, such that the numbers in each row and each column are in arithmetic progression. (Assume $n \geq 3$).
8. Determine all pairs (m, n) of positive integers for which

$$2^m + 3^n$$

is a perfect square.

9. Let $A_1 A_2 A_3 \ldots A_n$ be an n-sided regular polygon such that

$$\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}.$$

Determine n, the number of sides of the polynomial.

10. Determine all functions $f : \Re \setminus \{0, 1\} \longrightarrow \Re$ satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)},$$

where x is a real number different from 0 and 1.

(Here $\Re$ denotes the set of all real numbers.)