INMO–1989

Time : 3 hours

Attempt as many questions as you possibly can.

1. Prove that the polynomial

$$f(x) = x^4 + 26x^3 + 56x^2 + 78x + 1989$$

cannot be expressed as a product

$$f(x) = p(x)q(x)$$

where $p(x)$, $q(x)$ are both polynomials with integral coefficients and with degree ¡ 4.

2. Let a, b, c and d be any four real numbers, not all equal to zero. Prove that the roots of the polynomial

$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot all be real.

3. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots 541, 551\}$ having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

4. Determine with proof, all the positive integers n for which:

   (a) n is not the square of any integer; and

   (b) $[\sqrt{n}]$ divides n^2.

   (Notation : $[x]$ denotes the largest integer that is less than or equal to x).

5. For positive integers n, define $A(n)$ to be

$$\frac{(2n)!}{(n!)^2}.$$

Determine the sets of positive integers n for which

   (a) $A(n)$ is an even number,

   (b) $A(n)$ is a multiple of 4.

6. Triangle ABC has incenter I and the incircle touches BC, CA at D, E respectively. Let BI meet DE at G. Show that AG is perpendicular to BC.

7. Let A be one of the two points of intersection of two circles with centers X, Y respectively. The tangents at A to the two circles meet the circles again at B, C.

   Let a point P be located so that $PXAY$ is a parallelogram. Show that P is also the circumcenter of triangle ABC.