THEORETICAL PROBLEM No. 1
EVOLUTION OF THE EARTH-MOON SYSTEM
SOLUTIONS

1. Conservation of Angular Momentum

1a	$L_1 = I_E \omega_{E1} + I_{M1}\omega_{M1}$	0.2
1b	$L_2 = I_E \omega_2 + I_{M2}\omega_2$	0.2
1c	$I_E \omega_{E1} + I_{M1}\omega_{M1} = I_{M2}\omega_2 = L_1$	0.3

2. Final Separation and Angular Frequency of the Earth-Moon System.

2a	$\omega_2^2 D_2^3 = GM_E$	0.2
2b	$D_2 = \dfrac{L_1^2}{GM_E M_M^2}$	0.5
2c	$\omega_2 = \dfrac{G^2 M_E^2 M_M^3}{L_1^3}$	0.5
2d	The moment of inertia of the Earth will be the addition of the moment of inertia of a sphere with radius r_o and density ρ_o and of a sphere with radius r_i and density $\rho_i - \rho_o$: $I_E = \frac{2}{5}\frac{4\pi}{3}[r_o^5 \rho_o + r_i^5(\rho_i - \rho_o)]$.	0.5
2e	$I_E = \frac{2}{5}\frac{4\pi}{3}[r_o^5 \rho_o + r_i^5(\rho_i - \rho_o)] = 8.0\times10^{37}$ kg m^2	0.2
2f	$L_1 = I_E \omega_{E1} + I_{M1}\omega_{M1} = 3.4\times10^{34}$ kg m^2 s^{-1}	0.2
2g	$D_2 = 5.4\times10^8$ m, that is $D_2 = 1.4 D_1$	0.3
2h	$\omega_2 = 1.6\times10^{-6}$ s^{-1}, that is, a period of 46 days.	0.3
2i	Since $I_E \omega_2 = 1.3\times10^{32}$ kg m^2 s^{-1} and $I_{M2}\omega_2 = 3.4\times10^{34}$ kg m^2 s^{-1}, the approximation is justified since the final angular momentum of the Earth is 1/260 of that of the Moon.	0.2

3. How much is the Moon receding per year?

3a	Using the law of cosines, the magnitude of the force produced by the mass *m* closest to the Moon will be: $F_c = \dfrac{G\,m\,M_M}{D_1^2 + r_o^2 - 2\,D_1\,r_o\,\cos(\theta)}$	0.4

3b	Using the law of cosines, the magnitude of the force produced by the mass *m* farthest to the Moon will be: $F_f = \dfrac{G\,m\,M_M}{D_1^2 + r_o^2 + 2\,D_1\,r_o\,\cos(\theta)}$	0.4

3c	Using the law of sines, the torque will be $\tau_c = F_c \dfrac{\sin(\theta)\,r_0\,D_1}{[D_1^2 + r_o^2 - 2\,D_1\,r_o\,\cos(\theta)]^{1/2}} = \dfrac{G\,m\,M_M\,\sin(\theta)\,r_0\,D_1}{[D_1^2 + r_o^2 - 2\,D_1\,r_o\,\cos(\theta)]^{3/2}}$	0.4

3d	Using the law of sines, the torque will be $\tau_f = F_f \dfrac{\sin(\theta)\,r_0\,D_1}{[D_1^2 + r_o^2 + 2\,D_1\,r_o\,\cos(\theta)]^{1/2}} = \dfrac{G\,m\,M_M\,\sin(\theta)\,r_0\,D_1}{[D_1^2 + r_o^2 + 2\,D_1\,r_o\,\cos(\theta)]^{3/2}}$	0.4

3e	$\tau_c - \tau_f = G\,m\,M_M\,\sin(\theta)\,r_0\,D_1^{-2}\,\left(1 - \dfrac{3r_o^2}{2D_1^2} + \dfrac{3\,r_o\cos(\theta)}{D_1} - 1 + \dfrac{3r_o^2}{2D_1^2} + \dfrac{3\,r_o\cos(\theta)}{D_1}\right)$ $= \dfrac{6\,G\,m\,M_M\,r_o^2\,\sin(\theta)\cos(\theta)}{D_1^3}$	1.0

3f	$\tau = \dfrac{6\,G\,m\,M_M\,r_o^2\,\sin(\theta)\cos(\theta)}{D_1^3} = 4.1\times10^{16}\ \text{N m}$	0.5

3g	Since $\omega_{M1}^2 D_1^3 = GM_E$, we have that the angular momentum of the Moon is $I_{M1}\,\omega_{M1} = M_M\,D_1^2\left[\dfrac{G\,M_E}{D_1^3}\right]^{1/2} = M_M\,[D_1\,G\,M_E]^{1/2}$ The torque will be: $\tau = \dfrac{M_M\,[G\,M_E]^{1/2}\,\Delta(D_1^{1/2})}{\Delta t} = \dfrac{M_M\,[G\,M_E]^{1/2}\,\Delta D_1}{2[D_1]^{1/2}\,\Delta t}$ So, we have that $\Delta D_1 = \dfrac{2\,\tau\,\Delta t}{M_M}\left[\dfrac{D_1}{G\,M_E}\right]^{1/2}$ That for $\Delta t = 3.1\times10^7$ s = 1 year, gives $\Delta D_1 = 0.034$ m. This is the yearly increase in the Earth-Moon distance.	1.0

3h	We now use that $\tau = -\frac{I_E\,\Delta\omega_{E1}}{\Delta t}$ from where we get $\Delta\omega_{E1} = -\frac{\tau\,\Delta t}{I_E}$ that for $\Delta t = 3.1\times10^{7}$ s = 1 year gives $\Delta\omega_{E1} = -1.6\times10^{-14}$ s^{-1}. If P_E is the period of time considered, we have that: $\frac{\Delta P_E}{P_E} = -\frac{\Delta\omega_{E1}}{\omega_E}$ since $P_E = 1\ day = 8.64\times10^{4}$ s, we get $\Delta P_E = 1.9\times10^{-5}$ s. This is the amount of time that the day lengthens in a year.	1.0

4. Where is the energy going?

4a	The present total (rotational plus gravitational) energy of the system is: $E = \frac{1}{2} I_E\,\omega_{E1}^2 + \frac{1}{2} I_M\,\omega_{M1}^2 - \frac{G M_E M_M}{D_1}$. Using that $\omega_{M1}^2\,D_1^3 = G\,M_E$, we get $E = \frac{1}{2} I_E\,\omega_{E1}^2 - \frac{1}{2}\frac{G M_E M_M}{D_1}$	0.4

4b	$\Delta E = I_E\,\omega_{E1}\,\Delta\omega_{E1} + \frac{1}{2}\frac{G M_E M_M}{D_1^2}\,\Delta D_1$, that gives $\Delta E = -9.0\times10^{19}$ J	0.4

4c	$M_{water} = 4\pi r_o^2 \times h \times \rho_{water}$ kg = 2.6×10^{17} kg.	0.2

4d	$\Delta E_{water} = -g\,M_{water}\times 0.5\,m \times 2\,day^{-1}\times 365\,days \times 0.1 = -9.3\times10^{19}$ J. Then, the two energy estimates are comparable.	0.3