

RMO–1991

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1. Let P be an interior point of a triangle ABC and AP , BP , CP meet the sides BC , CA , AB in D , E , F respectively. Show that

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

2. If a , b , c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

3. A four-digit number has the following properties :

- (a) it is a perfect square,
- (b) its first two digits are equal to each other,
- (c) its last two digits are equal to each other.

Find all such four-digit numbers.

4. There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

- (a) remove an equal number of balls simultaneously from the urns, and
- (b) double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty. <http://www.iitscholars.com> <http://www.ntsescholars.com>

5. Take any point P_1 on the side BC of a triangle ABC and draw the following chain of lines : P_1P_2 parallel to AC (P_2 on AB), P_2P_3 parallel to BC , P_3P_4 parallel to AB , P_4P_5 parallel to CA , and P_5P_6 parallel to BC . Here P_2 , P_5 lie on AB ; P_3 , P_6 lie on CA ; and P_4 on BC . Show that P_6P_1 is parallel to AB .

6. Find all integer values of a such that the quadratic expression

$$(x + a)(x + 1991) + 1$$

can be factored as a product $(x + b)(x + c)$ where b and c are integers.

7. Prove that $n^4 + 4^n$ is composite for all integer values of n greater than 1.
8. The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.